EQUILIBRIUM MEASURES FOR THE HÉNON MAP AT THE FIRST BIFURCATION

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Abstract. We study the dynamics of strongly dissipative Hénon maps, at the first bifurcation parameter where the uniform hyperbolicity is destroyed by the formation of tangencies inside the limit set. We prove the existence of an equilibrium measure which minimizes the free energy associated with the non continuous potential $-t\log J^u$, where $t\in\mathbb{R}$ is in a certain interval of the form $(-\infty,t_0)$, where $t_0>1$ and J^u denotes the Jacobian in the unstable direction.

1. Introduction

An important problem in dynamics is to describe how horseshoes are destroyed. A process of destruction through homoclinic bifurcations is modeled by the Hénon family

$$f_a\colon (x,y)\mapsto(1-ax^2+\sqrt{b}y,\pm\sqrt{b}x),\quad 0<b\ll 1. \tag{1}$$

For all large a, the non-wandering set is a uniformly hyperbolic horseshoe [6]. As one decreases a, the stable and unstable directions get increasingly confused, and at last reaches a bifurcation parameter a^* near 2. The non-wandering set of f_a is a uniformly hyperbolic horseshoe for $a>a^*$, and (f_a) generically unfolds a quadratic tangency at $a=a^*$ [2, 3, 5, 8]. According to general theory of global bifurcations (for instance, see [16] and the references therein), a surprisingly rich array of complicated behaviors appear in the unfolding of the tangency. In this paper, instead of unfolding the tangency we study the dynamics of f_{a^*} from a viewpoint of ergodic theory and thermodynamic formalism. The dynamics of f_{a^*} is close to that of the uniformly hyperbolic horseshoe [2, 5, 8, 20], yet already exhibits some complexities shared by those f_a, $a<a^*$, and thus will provide an important insight into the bifurcation at a^*.

Another motivation for the study of f_{a^*} is to develop an ergodic theory for *non-attracting sets which are not uniformly hyperbolic*. In the rigorous study of dynamical systems, a great deal of effort has been devoted to the study of chaotic attractors. A statistical approach has been often taken, i.e., to look for nice invariant probability measures which statistically predict the asymptotic "fate" of positive Lebesgue measure sets of initial conditions. The non-wandering set of f_{a^*} behaves like a saddle, in that many orbits wander around it for a while due to its invariance, and eventually leave a neighborhood of it. Such non-attracting sets may be considered somewhat irrelevant, as they only concern transient behaviors. Although this point of view is justified by a wide variety of reasons, the study of non-attracting sets deserves our attention, because of their nontrivial influences on global dynamics. Moreover, important thermodynamic parameters relevant in this context, such as Hausdorff dimensions and escape rates, are not well-understood unless the uniform hyperbolicity is assumed.

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We state our setting and goals in more precise terms. Write f for f_{a^*}. Let K denote the set of initial points with bounded f-orbits, which is a compact set and coincides with the non-wandering set [5]. Let $\mathcal{M}(f)$ denote the space of all f-invariant Borel probability measures endowed with the topology of weak convergence. For a given potential $\varphi : K \to \mathbb{R}$ (the minus of) the free energy function $F_\varphi\colon \mathcal{M}(f) \to \mathbb{R}$ is given by $F_\varphi(\mu) = h(\mu) + \mu(\varphi)$, where $h(\mu)$ denotes the entropy of μ and $\mu(\varphi) = \int \varphi d\mu$. An *equilibrium measure* associated to the potential φ is a measure $\mu_\varphi \in \mathcal{M}(f)$ which maximize F_φ, i.e.

$$F_\varphi(\mu_\varphi) = \sup \{F_\varphi(\mu)\colon \mu \in \mathcal{M}(f)\}.$$

The existence and uniqueness of equilibrium measures depend upon the characteristics of the system and the potential. In our setting, the entropy map is upper semi-continuous (Corollary 3.2), and so equilibrium measures exist for any continuous potential and they are unique for a residual subset of continuous potentials [21, Corollary 9.15.1]. However, an important family of potentials is given by $\varphi_t = -t \log J^u$, where $t \in \mathbb{R}$ and J^u is the Jacobian in the *unstable direction*. Due to the presence of tangency, φ_t is merely bounded measurable. An equilibrium measure for the potential φ_t is called a *t-conformal measure*. Our goal is to prove the existence of t-conformal measures with t in a certain interval containing all negative t and some positive t.

At a point $z \in \mathbb{R}^2$, let $E^u(z)$ denote the one-dimensional subspace such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \|Df^{-n}|E^u(z)\| < 0. \tag{2}$$

Since f^{-1} expands area, $E^u(z)$ is unique when it makes sense. We call E^u an *unstable direction*. Let $J^u(z) = \|Df|E^u(z)\|$ and $\varphi_t = -t \log J^u$. Define

$$P(t) = \sup\{F_{\varphi_t}(\mu)\colon \mu \in \mathcal{M}(f)\}.$$

The *pressure function* $t \mapsto P(t)$ is convex, and so is continuous. Let

$$t_0 := \inf\{t \in \mathbb{R}\colon P(s) > -(s/3)\log(4-\varepsilon) \text{ for any } s < t\}.$$

Considering the two fixed saddles (see FIGURE 1) we can easliy see that $1 < t_0 < \infty$.

Theorem. *For any small $\varepsilon > 0$ there exists $b_0 > 0$ such that if $b < b_0$ and $t < t_0$, then there exists a t-conformal measure.*

Let us here mention some previous results closely related to ours which develop thermodynamics of systems *at the boundary of uniform hyperbolicity*. Makarov & Smirnov [13] studied rational maps on the Riemannian sphere for which every critical point in the Julia set is non-recurrent. Leplaideur & Rios [10, 11] proved the existence and uniqueness of t-conformal measures, for certain type 3 linear horseshoes in the plane (horseshoes with three symbols) with a single orbit of tangency studied in [17]. For their systems, Leplaideur [9] proved the analyticity of the pressure function. See Leplaideur, Oliveira & Rios [12] and Arbieto & Prudente [1] for results on partially hyperbolic horseshoes studied in [7].

The biggest difficulty in the proofs of the theorems is to handle the limit behavior of a sequence of Lyapunov exponents. For $\mu \in \mathcal{M}(f)$, let $\lambda^u(\mu) = \mu(\log J^u)$, which we call the *unstable Lyapunov exponent* of μ. Since $\log J^u$ is not continuous, the weak convergence $\mu_n \to \mu$ does not imply the convergence $\lambda^u(\mu_n) \to \lambda^u(\mu)$. We show that the unstable Lyapunov exponent is upper semi-continuous (Proposition 4.3). Hence, the existence of t-conformal measures for $t \le 0$ follows from the upper semi-continuity of F_{φ_t}. To show the existence for

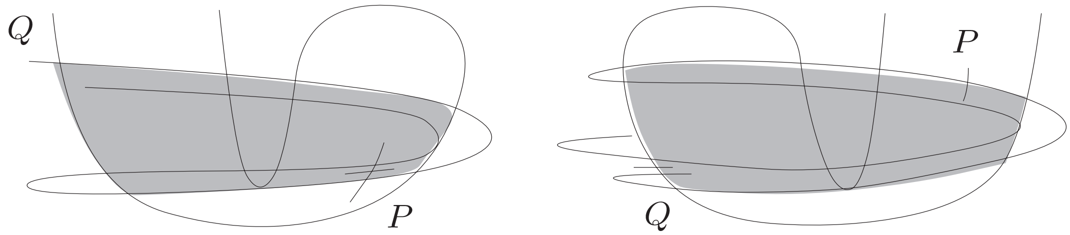

FIGURE 1. Manifold organization for $a = a^*$. There exist two hyperbolic fixed saddles P, Q near $(1/2, 0)$, $(-1, 0)$ correspondingly. In the orientation preserving case (left), $W^u(Q)$ meets $W^s(Q)$ tangentially. In the orientation reversing case (right), $W^u(P)$ meets $W^s(Q)$ tangentially. The shaded regions represent the region R (see Sect.2.1).

$t > 0$, we carefully analyze the sequence $\{\lambda^u(\mu_n)\}_n$. The analysis of the dynamics of f requires different arguments from [9, 10, 11], because the specific assumptions on their system do not hold for our f. Our approach follows the well-known line for Hénon-like systems [4, 15, 22].

The rest of this paper consists of three sections. In Sect.2 we study the dynamics of f. Our strategy is to introduce critical points in the spirit of Benedicks & Carleson [4], and use them to control the growth of derivatives in the unstable direction. The main technical result is stated in Proposition 2.6. In Sect.3 we show that the dynamics on K is semi-conjugated to the full shift on two symbols. Although this statement is not surprising, standard arguments do not work due to the presence of the tangency. In Sect.4 we combine the results in the previous sections and prove the theorems.

2. The dynamics

In this section we study the dynamics of f. In Sect.2.1 we state and prove basic geometric properties. Although the dynamics outside of a neighbourhood of the point of tangency are uniformly hyperbolic, returns to this neighbourhood is unavoidable. To control these returns, in Sect.2.2 we artificially introduce *critical points* following the idea of Benedicks & Carleson [4]. In Sect.2.3 we analyse the dynamics near the orbits of the critical points. In Sect.2.4 and Sect.2.5 we discuss how to associate critical points to generic orbits.

We deal with a few positive constants whose purposes are as follows:

- $\varepsilon, \delta, \tau, b$ are small constants chosen in this order: δ is used to define a critical region (see Sect.2.2); τ is used for various estimates and constructions in Sect.2; b concerns the dissipation of the map (1);
- four constants below are used for estimates of derivatives:

$$\sigma = 2 - \varepsilon,\ \lambda_1 = 4 - 2\varepsilon,\ \lambda_2 = 4 - \varepsilon,\ \lambda_3 = 4 + \varepsilon; \tag{3}$$

Any generic constant which is independent of ε, δ, τ b is simply denoted by C.

2.1. **Basic geometric properties.** Let P, Q denote the fixed saddles near $(1/2, 0)$ and $(-1, 0)$ correspondingly. If f preserves orientation, let $W^u = W^u(Q)$. If f reverses orientation, let $W^u = W^u(P)$. By a *rectangle* we mean any closed region bordered by two compact curves in W^u and two in the stable manifolds of P, Q. By an *unstable side* of a rectangle we mean any of the two boundary curves in W^u. A *stable side* is defined similarly.

Let R denote the rectangle as indicated in Figure 1. One of its unstable sides contains the point of tangency near $(0,0)$, which we denote by ζ_0. Let α_0^+ denote the stable side of R containing $f\zeta_0$ and let α_0^- denote the other stable side of R.

Let S denote the closed lenticular region bounded by the unstable side of R and the parabola in $W^s(Q)$ containing ζ_0. Points in the interior of S is mapped to the outside of R, and they never return to R under any positive iteration.

Let $\tilde\alpha_0$ denote the component of $W^s(P)\cap R$ containing P. Define a sequence $\{\tilde\alpha_k\}_{k\geq 1}$ of compact curves in $W^s(P)\cap R$ inductively as follows. Given $\tilde\alpha_{k-1}$, define $\tilde\alpha_k$ to be the one of both components of $f^{-1}\tilde\alpha_{n-1}\cap R$ which lies at the left of ζ_0.

For each $n\geq 0$ the set $f^{-2}\tilde\alpha_n\cap R$ consists of four curves, two of them at the left of ζ_0 and two at the right. Let α_{n+1}^- denote the one to the left of ζ_0 which is not $\tilde\alpha_{n+2}$. Among the two at the right of ζ_0, let α_{n+1}^+ denote the one which is at the left of the other.

Let Θ denote the rectangle bordered by α_1^-, α_1^+ and the unstable sides of R. More precisely, the curves obey the following diagram

$$\{\alpha_{n+1}^-,\alpha_{n+1}^+\}\xrightarrow{f^2}\tilde\alpha_n\xrightarrow{f}\tilde\alpha_{n-1}\xrightarrow{f}\tilde\alpha_{n-2}\xrightarrow{f}\cdots\xrightarrow{f}\tilde\alpha_1=\alpha_1^-\xrightarrow{f}\tilde\alpha_0=\alpha_1^+.$$

We now investigate the geometry of W^u. A specific feature of the map f which follows from the results in [20] is that "folds" in W^u do not enter Θ. Hence the following holds. By a $C^2(b)$*-curve* we mean a closed curve, not necessarily in W^u, for which the slopes of its tangent directions are $\leq\sqrt{b}$ and the curvature is everywhere $\leq\sqrt{b}$.

Lemma 2.1. [20, Section 4] *Any component of $\Theta\cap W^u$ is a $C^2(b)$-curve with endpoints in α_1^-,α_1^+.*

For $k\geq 0$, let $\Delta_k=\Theta\cap f^kR$. Observe that Δ_k has 2^k components each of which is a rectangle, and by Lemma 2.1, the unstable sides of it are $C^2(b)$-curves. Also observe that Δ_k is related to Δ_{k-1} as follows: let $\mathcal{Q}_{k-1}$ denote any component of Δ_{k-1}. Then $\mathcal{Q}_{k-1}\cap f^kR$ has two components, each of which is a component of Δ_k.

Lemma 2.2. *For $k=0,1,\dots$ and for each component $\mathcal{Q}_k$ of Δ_k, the Hausdorff distance between its unstable sides is $\mathcal{O}(b^{\frac{k}{2}})$.*

Proof. We argue by induction on k. Assume the statement for $0\leq k<j$. We regard the unstable sides of $\mathcal{Q}_j$ as graphs of functions γ_1, γ_2 defined on an interval I. Let $L(x)=|\gamma_1(x)-\gamma_2(x)|$. Since $\mathcal{Q}_j$ is contained in a component of Δ_{j-1}, the assumption of induction gives $L^{\frac12}(x)\leq(Cb)^{\frac{j-1}{4}}<\text{length}(I)$. Moreover $|\gamma_1'(x)-\gamma_2'(x)|\leq L^{\frac12}(x)$ holds, since γ is C^2 and so otherwise γ_1 would intersect γ_2. By this and the definition of admissible curves, $L(y)\geq L(x)-(L^{\frac12}(x)-C\sqrt{b}|x-y|)|x-y|$ holds for $x,y\in I$, which is $\geq L(x)/2$ provided $|x-y|\leq L^{\frac23}(x)$. Hence, $\text{area}(\mathcal{Q}_j)\geq L^{\frac53}(x)/2$ holds. If $L(x)\geq b^{\frac{j}{2}}$, then $\text{area}(\mathcal{Q}_j)\geq b^{\frac{5j}{6}}/2$, which yields a contradiction to $\text{area}(\mathcal{Q}_j)<\text{area}(f^jR)\leq(Cb)^j$. $\square$

2.2. **Critical points.** Define

$$I(\delta)=(-\delta,\delta)\times(-\sqrt{b},\sqrt{b}).$$

Observe that, for any given $\delta>0$, the point ζ_0 of tangency is contained in $I(\delta)$ provided b is sufficiently small. The next lemma readily follows from the fact that f is regarded as a perturbation of the Chebyshev quadratic $x\mapsto 1-2x^2$ which in turn is smoothly conjugate to the tent map. We say a nonzero tangent vector v is *b-horizontal* if $\text{slope}(v)\leq\sqrt{b}$.

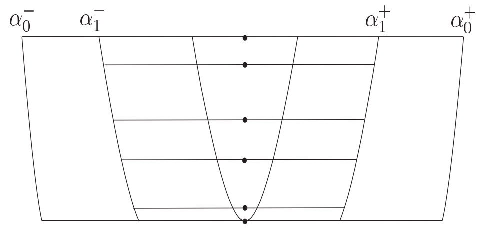

FIGURE 2. critical points on W^u

Lemma 2.3. *For any $\varepsilon > 0$, $\delta > 0$ there exists $b_0 = b_0(\varepsilon, \delta) > 0$ such that the following holds provided $b < b_0$:*

(a) *if $n \geq 1$ and $z \in R$ is such that $z, fz, \dots, f^{n-1}z \notin I(\delta)$ and $f^n z \in I(\delta)$, then for any b-horizontal vector v at z, $Df^n(z)v$ is b-horizontal and $\|Df^n(z)v\| \geq \sigma^n \|v\|$;*

(b) *if $z \in [-2, 2]^2 \setminus \Theta$, then for any b-horizontal vector v at z, $Df(z)v$ is b-horizontal and $\|Df(z)v\| \geq \sigma \|v\|$.*

To recover the loss of hyperbolicity due to returns to the inside of $I(\delta)$ we mimic the strategy of Benedicks & Carleson [4]: we develop a binding argument relative to *critical points*.

From the hyperbolicity of the saddle Q it follows that (use the Center Manifold Theorem [18] for the tangent bundle map) there exists a neighborhood U of $\alpha_0^- \cup \alpha_0^+$ and a foliation $\mathcal{F}^s$ of U such that:

(F1) $\mathcal{F}^s(Q)$, the leaf of $\mathcal{F}^s$ containing Q, contains α_0^-;
(F2) if $z, fz \in U$, then $f(\mathcal{F}^s(z)) \subset \mathcal{F}^s(fz)$;
(F3) Let $e^s(z)$ denote the unit vector in $T_z\mathcal{F}^s(z)$ with the positive second component. Then: $z \to e^s(z)$ is C^1 and $\|Dfe^s(z)\| \leq Cb$, $\|\frac{\partial}{\partial z}e^s(z)\| \leq C$;
(F4) If $z, fz \in U$, then $\text{slope}(e^s(z)) \geq C/\sqrt{b}$.

We call $\mathcal{F}^s$ a *stable foliation* on U. From (F1), (F2) and $f\alpha_0^+ \subset \alpha_0^-$ it follows that there is a leaf of $\mathcal{F}^s$ which contains α_0^+. (F4) can be checked by contradiction: if it were false, then $\text{slope}(e^s(fz)) \ll 1$.

Definition 2.4. We say $\zeta \in W^u$ is a *critical point* if $f\zeta \in U$ and $T_{f\zeta}W^u = T_{f\zeta}\mathcal{F}(f\zeta)$.

It follows from [20] that any component of $\Theta \cap W^u$ admits a unique critical point, and it is contained in S. Then it follows that:

- the non wandering set K does not contain any critical point other than ζ_0;
- any other critical point is mapped by f to the outside of R, and then escapes to infinity under positive iteration.

Pieces of critical orbits far away from R are irrelevant. We say a critical point ζ is *non escaping up to time n* if $f^i\zeta \in U$ holds for every $1 \leq i \leq n$.

Let ζ be a critical point non escaping up to time n. For $i \geq 1$ let $w_i(\zeta) = Df^{i-1}(f\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Since $f\zeta, \dots, f^{i-1}\zeta \notin \text{int}R$ we have

$$\text{slope}(w_i(\zeta)) \leq \sqrt{b},$$

and for all $0 < i < j$,

$$\|w_j(\zeta)\| \geq \lambda_2^{j-i}\|w_i(\zeta)\|. \tag{4}$$

Choose a constant $\tau > 0$ so that U contains the 6τ-neighborhood of R and $\frac{100\tau}{\sigma} \leq \log 2$. For $n \geq 1$, define

$$D_n(\zeta) = \tau \left[\sum_{i=1}^{n} d_i^{-1}(\zeta) \right]^{-1}, \quad \text{where} \quad d_i(\zeta) = \frac{\|w_{i+1}(\zeta)\|}{\|w_i(\zeta)\|^2}. \tag{5}$$

Lemma 2.5. *There exists $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$, then*
(a) $(\lambda_3 + \varepsilon)^{-n} \leq D_n(\zeta) \leq \lambda_2^{-n}$;
(b) $\tau/5 \leq \|w_n(\zeta)\| D_n(\zeta) \leq 5\tau$.

Proof. (4) yields

$$(\lambda_3 + \varepsilon)^{-n} \leq \frac{\tau}{n} \cdot \lambda_2 \lambda_3^{-n+1} \leq \frac{\tau}{n} \cdot \min_{1 \leq i \leq n} d_i(\zeta) \leq D_n(\zeta) \leq \tau d_n(\zeta) \leq 5\tau \lambda_2^{-n+1} \leq \lambda_2^{-n}.$$

The first inequality holds for sufficiently large n depending only on ε. As for (b) we have

$$\|w_n(\zeta)\| D_n(\zeta) < \tau \|w_n(\zeta)\| d_n(\zeta) = \tau \frac{\|w_{n+1}(\zeta)\|}{\|w_n(\zeta)\|} \leq 5\tau.$$

For the lower estimate, (4) yields

$$\frac{1}{\|w_n(\zeta)\| D_n(\zeta)} = \frac{1}{\tau} \sum_{i=1}^{n} \frac{\|w_i(\zeta)\|}{\|w_n(\zeta)\|} \frac{\|w_i(\zeta)\|}{\|w_{i+1}(\zeta)\|} \leq \frac{1}{\tau} \sum_{i=1}^{n} \lambda_2^{-(n-i+1)} \leq \frac{5}{\tau}.$$

□

2.3. **Recovering hyperbolicity.** In this subsection we assume ζ is a critical point, and γ is a $C^2(b)$-curve in $I(\delta)$ which is tangent to W^u at ζ. Consider the leaf $\mathcal{F}^s(f\zeta)$ of the stable foliation $\mathcal{F}^s$ through $f\zeta$. The leaf may be expressed as a graph of a smooth function: there exists an open interval J and a smooth function $y \mapsto x(y)$ on J such that

$$\mathcal{F}^s(f\zeta) = \{(x(y), y) \colon y \in J\}.$$

Let $z \in \gamma$. We say $p = p(z)$ is a *bound period of* z if

$$fz \in \{(x, y) \colon D_p(\zeta) < |x - x(y)| \leq D_{p-1}(\zeta),\ y \in J\}. \tag{6}$$

If ζ is non escaping up to time p, then define a *fold period* $q = q(z)$ of z by

$$q = \min\left\{1 \leq i < p \colon |\zeta - z|^\beta \cdot \|w_{j+1}(\zeta)\| \geq 1 \ \text{ for every } \ i \leq j < p\right\}, \tag{7}$$

where

$$\beta = 2/\log(1/b). \tag{8}$$

(6) (8) yield $|\zeta - z|^\beta \cdot \|w_p(\zeta)\| \geq 1$, and so q is well-defined.

The purposes of these two periods are as follows: the fold period is used to restore the slopes of iterated tangent vectors to b-horizontal; the bound period is used to recover an expansion of derivatives. Let us agree that, for two positive numbers a, b, $a \approx b$ indicates $1/C \leq a/b \leq C$ for some $C \geq 1$.

Proposition 2.6. *Let ζ be a critical point and γ a $C^2(b)$-curve in $I(\delta)$ which is tangent to $E^u(\zeta)$. If $z \in \gamma$ and p, q are the corresponding bound and fold periods, then:*
(a) $\log |\zeta - z|^{-\frac{2}{\log(\lambda_3 + 2\varepsilon)}} \leq p \leq \log |\zeta - z|^{-\frac{3}{\log \lambda_2}}$;
(b) $\log |\zeta - z|^{-\frac{\beta}{\log \lambda_3}} \leq q \leq \log |\zeta - z|^{-\frac{2\beta}{\log \lambda_2}}$.

Let $v(z)$ denote any unit vector tangent to γ at z. Then:

(c) $\|Df^iv(z)\| \approx |\zeta - z| \cdot \|w_i(\zeta)\|$ *for every* $q < i \leq p$;
(d) $\|Df^iv(z)\| < 1$ *for every* $1 \leq i < q$;
(e) $\|Df^pv(z)\| \geq \lambda_1^{\frac{p}{2}}$;
(f) $\mathrm{slope}(Df^pv(z)) \leq \sqrt{b}$.

A proof of this proposition follows the well-known line [4, 15, 22] that is now well-understood. We split $Dfv(z)$ into $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$-component and $e^s(fz)$-component, and iterate them separately. The latter is contracted exponentially, and the former copies the growth of $w_1(\zeta), \ldots, w_p(\zeta)$, and thus gains an expansion. The contracted component is eventually dominated by the expanded one, and as a result the desired estimates holds.

Proof of Proposition 2.6. We divide the proof into three steps.

Step 1(Bounded distortion). We first establish a bounded distortion in the strip

$$\{(x,y)\colon |x - x(y)| \leq D_{p-1}(\zeta), y \in J\}\,. \tag{9}$$

Lemma 2.7. *Let* $(x(y_0), y_0) \in \mathcal{F}^s(f\zeta)$, *and let* γ_0 *be the horizontal segment of the form* $\gamma_0 = \{(x, y_0)\colon |x - x(y_0)| \leq D_{p-1}(\zeta)\}$. *Then:*
(a) *for all* $\xi, \eta \in \gamma_0$ *and every* $1 \leq i < p$, $\|Df^i(\xi)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \leq 2 \cdot \|Df^i(\eta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|$;
(b) *for every* $1 \leq i < p$, $f^i\gamma_0$ *is a* $C^2(b)$*-curve and* $\mathrm{length}(f^i\gamma_0) \leq 20\tau$.

Proof. These estimates would hold if for all $0 \leq j < p-1$ we have

$$f^j\gamma_0 \subset [-2,2]^2 \setminus \Theta, \quad \mathrm{length}(f^j\gamma_0) \leq 20 d_{j+1}^{-1}(\zeta) D_{p-1}(\zeta) \leq 20\tau. \tag{10}$$

Indeed, let $1 \leq i < p$. Summing (10) over all $j = 0, 1, \ldots, i-1$ yields

$$\begin{aligned}\log \frac{\|Df^i(\xi)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}{\|Df^i(\eta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|} &= \sum_{j=0}^{i-1} \log \frac{\|Df^j(\xi)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}{\|Df^j(\eta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|} \leq \frac{1}{\sigma}\sum_{j=0}^{i-1} \|Df(f^j\xi)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - Df(f^j\eta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \\ &\leq \frac{5}{\sigma}\sum_{j=0}^{i-1} \mathrm{length}(f^j\gamma_0) \leq \frac{100\tau}{\sigma} \leq \log 2.\end{aligned}$$

We prove (10) by induction on j. It is immediate to check it for $j = 0$. Let $k > 0$ and assume (10) for every $0 \leq j < k$. Then, from the form of our map (1), $f^k\gamma_0$ is a $C^2(b)$-curve. Summing the inequality in (10) over all $0 \leq j < k$ yields $\|Df^k(\xi)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \leq 2 \cdot \|Df^k(\eta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|$ for all $\xi, \eta \in \gamma_0$. By a result of [15, Section 6], $\|Df^k(z_0)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \leq 2 \cdot \|Df^k(f\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|$, where $z_0 = (x(y_0), y_0)$. Hence

$$\begin{aligned}\mathrm{length}(f^k\gamma_0) &\leq 4\|w_{k+1}(\zeta)\| D_{p-1}(\zeta) = 4 d_{k+1}^{-1}(\zeta) D_{p-1}(\zeta) \frac{\|w_{k+2}(\zeta)\|}{\|w_{k+1}(\zeta)\|} \\ &\leq 20 d_{k+1}^{-1}(\zeta) D_{p-1}(\zeta) \leq 20\tau.\end{aligned}$$

Since ζ is non escaping up to time p and $p > k$, $f^k\zeta \in U$ and thus $f^k\gamma_0 \subset [-2,2]^2 \setminus \Theta$ holds. Hence (10) holds for $j = k$. □

Now, take arbitrary two points ξ_1, ξ_2 in the strip (9), and denote by η_σ the point of $\mathcal{F}^s(f\zeta)$ with the same y-coordinate as that of ξ_σ ($\sigma = 1, 2$). By the result of [[15], Section 6], $\|Df^i(\eta_1)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \leq 2 \cdot \|Df^i(\eta_2)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|$ holds for every $1 \leq i < p$. This and Lemma 2.7(a) yield

$$\|Df^i(\xi_1)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \leq 8 \cdot \|Df^i(\xi_2)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \quad \text{for every } 1 \leq i < p. \tag{11}$$

Step 2(Proofs of (a),(b)*).* Split

$$Df v(z) = A_0 \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B_0 \cdot e^s(f\zeta).$$

By [[19] Lemma 2.2],

$$|A_0| \approx |\zeta - z| \text{ and } |B_0| \le C\sqrt{b}. \tag{12}$$

For a point r near $f\zeta$, write $r = f\zeta + \xi(r)w_1(\zeta)^T + \eta(r)e^s(f\zeta)^T$, where T denotes the transpose. The integrations of the inequalities in (12) along γ from ζ to z give

$$|\xi(fz)| \approx |\zeta - z|^2 \text{ and } |\eta(fz)| \le C\sqrt{b}|\zeta - z|. \tag{13}$$

Write $fz = (x_0, y_0)$ and $f\zeta = (x_1, y_1)$. Since $f\gamma$ is tangent to $\mathcal{F}^s(f\zeta)$ at $f\zeta$ we have $\frac{d\xi(x(y),y)}{dy}(y_1) = 0$. (F1) gives $\left|\frac{d^2\xi(x(y),y)}{dy^2}\right| \le C$. Then

$$|\xi(x(y_0), y_0)| \le C|y_0 - y_1|^2.$$

(13) gives

$$|y_0 - y_1|^2 \le C|\eta(fz)|^2 \le Cb|\zeta - z|^2.$$

Since $|x_0 - x(y_0)| = |\xi(fz) - \xi(x(y_0), y_0)|$, the above two inequalities and (13) yield

$$|x_0 - x(y_0)| \approx |\zeta - z|^2. \tag{14}$$

Using (6) (14) and Lemma 2.5(a) we have

$$|\zeta - z|^2 \le C \cdot D_{p-1}(\zeta) \le C \cdot \lambda_2^{-p} \tag{15}$$

and

$$|\zeta - z|^2 \ge C \cdot D_p(\zeta) \ge C \cdot (\lambda_3 + \varepsilon)^{-p}.$$

Taking logs and then rearranging the results we obtain (a).

(4) and the definition of q give

$$\lambda_2^{q-1} \le \|w_q(\zeta)\| < |\zeta - z|^{-\beta}$$

and

$$\lambda_3^q \ge \|w_{q+1}(\zeta)\| \ge |\zeta - z|^{-\beta}.$$

Taking logs of these and then rearranging the result yields (b).

Step 3(Proofs of (c)-(f)*).* Split

$$Df v(z) = A \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B \cdot e^s(fz),$$

and write

$$e^s(fz) = \begin{pmatrix}\cos\theta(z)\\ \sin\theta(z)\end{pmatrix} \text{ and } \rho \cdot Df v(z) = \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix},$$

where $\theta, \psi \in [0, \pi)$ and $\rho > 0$ is the normalizing constant. (12) implies $|\theta(\zeta) - \psi| \approx \rho^{-1}|\zeta - z| \gg |\zeta - z|$. (F1) gives $|\theta(\zeta) - \theta(z)| \le C|\zeta - z| \ll |\theta(\zeta) - \psi|$, which implies $|\theta(z) - \psi| \approx |\theta(\zeta) - \psi|$. Hence

$$|A| \approx \rho|\theta(z) - \psi| \approx \rho|\theta(\zeta) - \psi| \approx |\zeta - z|. \tag{16}$$

Using this and the bounded distortion (11) we have

$$|A| \cdot \|Df^{i-1}(fz) \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \approx |\zeta - z| \cdot \|w_i(\zeta)\| \ge C|\zeta - z|^{1-\beta}, \tag{17}$$

where the second inequality follows from the definition of q. We also have

$$|B| \cdot \|Df^{i-1}e^s(fz)\| \leq (Cb)^{i-1} \leq (Cb)^q \leq |\zeta - z|^{\frac{3}{2}}. \tag{18}$$

For the last inequality in (18) we have used the lower estimate of q, the definition of β and Proposition 2.6(b). For the first inequality in (18) we have used the invariance (F2) of the stable foliation $\mathcal{F}^s$ and the contraction in (F3) for the iterates of z. This argument is justified by the following claim. Recall that U is the domain where e^s makes sense.

Claim 2.8. *For every $1 \leq i \leq p$, $z, fz, f^2z, \ldots, f^{i-1}z \in U$.*

Proof. Use Lemma 2.7(b) and the fact that $\tau \ll 1$. □

(17) (18) yield (c).

Let $i \leq q$. The definition of q and $\|w_q(\zeta)\| \geq \|w_i(\zeta)\|$ give

$$|A| \cdot \|Df^{i-1}(fz)\| \leq |\zeta - z| \cdot \|w_i(\zeta)\| \leq |\zeta - z|^{1-\beta} \ll 1.$$

This and (18) yield (d).

As for (e), we use (6) (14) and the second inequality of Lemma 2.5(a) to obtain $|\zeta - z|^{-1} \geq C\lambda_2^{\frac{p}{2}}$. In addition, using (14) and the first inequality of Lemma 2.5(b) we have $\|w_p(\zeta)\| \cdot |\zeta - z|^2 \geq C\|w_p(\zeta)\| \cdot D_p(\zeta) \geq C$. Hence

$$\|Df^pv(z)\| \geq C\|w_p(\zeta)\| \cdot |\zeta - z| \geq C|\zeta - z|^{-1} \geq C\lambda_2^{\frac{p}{2}} \geq \lambda_1^{\frac{p}{2}},$$

where the last inequality holds provided δ is sufficiently small. (f) follows from (c). □

2.4. **Unstable leaves.** Let $\tilde{\Gamma}^u$ denote the collection of $C^2(b)$-curves in W^u with endpoints in the stable sides of Θ. Let

$$\Gamma^u = \{\gamma^u \colon \gamma^u \text{ is the pointwise limit of a sequence in } \tilde{\Gamma}^u\}.$$

Any curve in Γ^u is called an *unstable leaf.* By the $C^2(b)$-property, the pointwise convergence is equivalent to the uniform convergence. Since two distinct curves in $\tilde{\Gamma}^u$ do not intersect each other, the uniform convergence is equivalent to the C^1 convergence. Hence, any unstable leaf is a C^1 curve with endpoints in Θ and the slopes of its tangent directions are $\leq \sqrt{b}$. Let $\mathcal{W}^u$ denote the union of all unstable leaves.

Lemma 2.9. $\Theta \cap K \subset \mathcal{W}^u$.

Proof. Let $z \in \Theta \cap K$. Then there exists an arbitrarily large integer k such that $f^{-k}z \notin I(\delta)$. Since $z \in K$, $f^{-k}z \in R$. Hence, $z \in \Delta_k$ holds. Since k can be made arbitrarily large, from Lemma 2.2 z is accumulated by curves in $\tilde{\Gamma}^u$. Hence z is contained in an unstable leaf. □

2.5. **Bound/free structure.** Let $z \in K \cap I(\delta)$. To the forward orbit of z we associate inductively a sequence of integers $0 =: n_0 < n_0 + p_0 < n_1 < n_1 + p_1 < n_2 < n_2 + p_2 < \cdots$, and then introduce useful terminologies along the way. To this end we need a couple of lemmas.

Lemma 2.10. *If $z \in K \cap I(\delta)$, then there exists a critical point ζ and a $C^2(b)$-curve γ which is tangent to both $T_\zeta W^u$ and $E^u(z)$.*

Proof. Since $z \in K \cap I(\delta)$, by Lemma 2.9 it is accumulated by $C^2(b)$-curves in W^u with endpoints in α_1^-, α_1^+, each of which admits a critical points. Hence the claim follows. □

Lemma 2.11. *Let ζ be a critical point non escaping up to time n and $f^{n+1}\zeta \notin U$. If $z \in I(\delta)$ and $p(z) > n$, then $f^{n+1}z \notin R$.*

Proof. By Lemma 2.7(b) we have $|f^n z - f^n\zeta| \leq 20\tau + (Cb)^{n-1}$, and thus $|f^{n+1}z - f^{n+1}\zeta| \leq 5(\tau + (Cb)^{n-1})$. Since $\tau \ll 1$, this implies $f^{n+1}z \notin R$. □

Given n_i with $f^{n_i}z \in I(\delta)$, take a critical point ζ and a $C^2(b)$-curve γ which is tangent to both $T_\zeta W^u$ and $E^u(f^{n_i}z)$. Let p_i denote the bound period of $f^{n_i}z$ given by the definition in Sect.2.3 applied to (ζ, γ). Then ζ is non escaping up to time p_i, for otherwise Lemma 2.11 yields $f^{n_i+p_i}z \notin R$, a contradiction to $z \in K$.

Let n_{i+1} denote the next return time of the orbit of z to $I(\delta)$ after $n_i + p_i$. Then Lemma 2.11 applies to $f^{n_{i+1}}z$. A recursive argument allows us to decompose the forward orbit of z into segments corresponding to time intervals $(n_i, n_i + p_i)$ and $[n_i + p_i, n_{i+1}]$, during which we describe the points in the orbit of z as being "bound" and "free" states respectively. Each n_i is called a *free return time.*

Let us record the following derivative estimates:

$$\|Df^{p_i}|E^u(f^{n_i}z)\| \geq \lambda_1^{\frac{p_i}{2}} \quad \text{and} \quad \|Df^{n_{i+1}-n_i-p_i}|E^u(f^{n_i+p_i}z)\| \geq \sigma^{n_{i+1}-n_i-p_i}. \tag{19}$$

The first one is a consequence of Proposition 2.6. The second one follows from Lemma 2.3 and the fact that $E^u(f^{n_i+p_i}z)$ is spanned by a b-horizontal vector, which in turn follows from Proposition 2.6(f).

3. Symbolic coding

In this section we show that $f|K$ is semi-conjugate to the full shift on two symbols. As a corollary we obtain an upper semi-continuity of entropy. In Sect.3.1 we give precise statements of main results in this section. In Sect.3.2 we introduce some relevant definitions, and in Sect.3.3 we construct the semi-conjugacy.

3.1. **Upper semi-continuity of entropy.** The region $R \setminus \mathrm{int}S$ consists of two rectangles, intersecting each other only at ζ_0. Let R_0 denote the one at the left of ζ_0 and let R_1 denote the one at the right. Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ denote the shift space endowed with the product topology of the discrete topology in $\{0,1\}$, and $\sigma\colon \Sigma_2 \circlearrowleft$ the left shift. Let $\pi\colon \Sigma_2 \to K$ denote the coding map, namely, for $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Sigma_2$, let

$$\pi(\omega) = \{x \in K\colon f^n x \in R_{\omega_n}\ \forall n \in \mathbb{Z}\}.$$

Proposition 3.1. *For any $\omega \in \Sigma_2$, $\pi(\omega)$ consists of a single point. In addition, π is surjective, continuous, 1-1 except on $\bigcup_{i=-\infty}^{\infty} f^i\zeta_0$ where it is 2-1. It gives a semi-conjugacy $\pi \circ \sigma = f \circ \pi$.*

Corollary 3.2. *The entropy map $\mu \in \mathcal{M}(f) \to h(\mu)$ is upper semi-continuous. In particular, any continuous function admits an equilibrium measure. Moreover, for a residual set of continuous potential functions this equilibrium measure is unique.*

Proof. Let $\mathcal{M}(\sigma)$ denote the space of σ-invariant Borel probability measures endowed with the topology of weak convergence. The push-forward $\pi_*\colon \mathcal{M}(\sigma) \to \mathcal{M}(f)$ is a continuous map from a compact space to a Hausdorff space. To show that π_* is bijective, we use the following, the proof of which is left as an exercise.

Claim 3.3. *Let X_i be a topological space and $\mathcal{B}_i$ its Borel σ-algebra, $i = 1, 2$. Let $h\colon X_1 \to X_2$ be a bijective map which sends open sets to Borel sets. Then h^{-1} is measurable.*

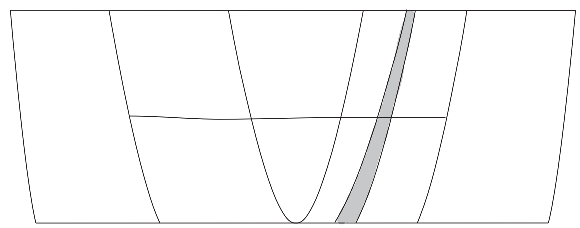

FIGURE 3.

Let $K_0 = K \setminus \pi^{-1} \bigcup_{n=-\infty}^{\infty} f^n \zeta_0$ and $\pi_0 = \pi | K_0$. Since π_0 is bijective and sends open sets to measurable sets, by Claim 3.3 it is a measurable bijection, and thus the pull-back π_0^* is well-defined. Since any $\nu \in \mathcal{M}(f)$ gives full weight to K_0, $\pi_0^*(\nu) \in \mathcal{M}(\sigma)$. Hence π_* is bijective. In particular π_* is a homeomorphism, and the inverse is π_0. Then the existence follows directly from the upper semi-continuity of the entropy map of σ. The uniqueness follows as in [21, Corollary 9.15.1]. □

3.2. **Relevant definitions.** By an *s-rectangle* we mean a rectangle in R whose unstable sides belong to the unstable sides of R. A *u-rectangle* is a rectangle in R whose stable sides belong to the stable sides of R. Let $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Sigma_2$, and write $\omega = (\omega^-, \omega^+) \in \Sigma_2$, where $\omega^- = (\omega_n)_{n<0}$ and $\omega^+ = (\omega_n)_{n \geq 0}$. For the cylinder set $[\omega_{-n} \cdots \omega_{-1}]$ in Σ_2, $\pi[\omega_{-n} \cdots \omega_{-1}]$ is a u-rectangle. Similarly, $\pi[\omega_0 \cdots \omega_n]$ is an s-rectangle in R_{ω_0}. Let $V^u(\omega^-) = \bigcap_{n<0} \pi[\omega_{-n} \cdots \omega_{-1}]$ and $V^s(\omega^+) = \bigcap_{n \geq 0} \pi[\omega_0 \cdots \omega_n]$. We have $\pi(\omega) = V^u(\omega^-) \cap V^s(\omega^+)$.

By a *u-curve* we mean the unstable sides of Θ, or else a compact C^1 curve in Θ which connects the stable sides of Θ, and intersects the boundary of Θ at no point other than its endpoints. An *s-curve* is defined similarly.

For an u-curve γ, let $\Theta(\gamma)$ denote the closed region bordered by γ, the unstable side of Θ containing ζ_0 and the stable sides of Θ. If γ is one of the unstable sides of Θ, then let $\Theta(\gamma) = \Theta$.

3.3. **Proof of Proposition 3.1.** A bulk of the proof is to show that $\pi(\omega)$ is one point, for any $\omega \in \Sigma_2$. In the coding of the uniformly hyperbolic horseshoe, one considers two families of stable and unstable strips (s/u-rectangles in our terms) and show that their boundary curves converge to curves, intersecting each other exactly at one point. In our situation, due to the presence of tangency, the convergence of boundaries of s-rectangles is not clear. To circumvent this point, we take advantage of the first bifurcation situation.

Lemma 3.4. *If $\Theta \cap \pi(\omega) \neq \emptyset$, then it is one point.*

Proof. Lemma 2.2 implies that $\Theta \cap V^u(\omega^-)$ is a C^1 limit of the unstable sides of $\pi[\omega_{-n} \cdots \omega_{-1}]$, and hence is an u-curve. Meanwhile, since $\Theta \cap \pi(\omega) \neq \emptyset$, one of the stable sides of $\pi[\omega_0 \cdots \omega_n]$ is contained in Θ. Let $\partial^s \pi[\omega_0 \cdots \omega_n]$ denote any stable side of $\pi[\omega_0 \cdots \omega_n]$. Since $W^s(P)$ does not intersect itself, either $\partial^s \pi[\omega_0 \cdots \omega_n] \subset \Theta$ or $\subset R \setminus \Theta$. In the former case, $\partial^s \pi[\omega_0 \cdots \omega_n]$ intersects $\Theta \cap V^u(\omega^-)$. We do not claim the uniqueness of this intersection, and argue as

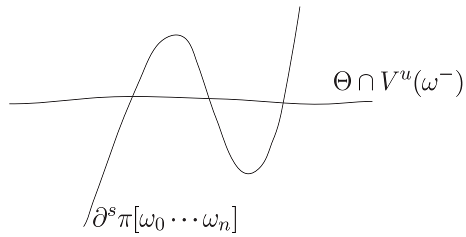

FIGURE 4. situation eliminated by Sublemma 3.5

follows. The next sublemma indicates that if $\partial^s\pi[\omega_0 \cdots \omega_n] \subset \Theta$, then it does not wind around $\Theta \cap V^u(\omega^-)$ (See Figure 4).

Sublemma 3.5. *If $\partial^s\pi[\omega_0 \cdots \omega_n] \subset \Theta$, then $\partial^s\pi[\omega_0 \cdots \omega_n] \cap \Theta(V^u(\omega^-))$ is connected.*

Proof. Suppose this intersection is not connected. By Lemma 2.2, the unstable sides of $\pi[\omega_{-n} \cdots \omega_{-1}] \cap \Theta$ converge in C^1 to $\Theta \cap V^u(\omega^-)$. Hence, it is possible to choose an integer $m > 0$ and an unstable side $\Theta(\partial^u\pi[\omega_{-m} \cdots \omega_{-1}])$ of $\pi[\omega_{-m} \cdots \omega_{-1}] \cap \Theta$ such that $\partial^s\pi[\omega_0 \cdots \omega_n] \cap \Theta(\partial^u\pi[\omega_{-m} \cdots \omega_{-1}])$ is not connected.

Consider the *continuations* $(\cdot)$ of these two curves for $a \geq a^*$. Since a^* is a first bifurcation, f_a for $a > a^*$ is Smale's horseshoe map. Hence, $\partial^s\pi[\omega_0 \cdots \omega_n](a) \cap \Theta(\partial^u\pi[\omega_{-m} \cdots \omega_{-1}])(a)$ has to be connected. By the continuous parameter dependence of invariant manifolds, there must come a parameter $a_0 > a^*$ such that $\partial^s\pi[\omega_0 \cdots \omega_n](a_0)$ and $\partial^u\pi[\omega_{-m} \cdots \omega_{-1}](a_0)$ meet each other tangentially. This is a contradiction to that a^* is a first bifurcation. $\square$

By Sublemma 3.5, $\Theta \cap V^u(\omega^-) \cap \pi[\omega_0 \cdots \omega_n]$ is a closed curve in $\Theta \cap V^u(\omega^-)$, and is a nested sequence. Hence, $\Theta \cap V^u(\omega^-) \cap V^s(\omega^+) = \Theta \cap \pi(\omega)$ is a point, or a closed curve. To finish, we eliminate the latter case by contradiction.

Let $\gamma = \Theta \cap \pi(\omega)$ and suppose that γ is a closed curve, not a point. Since γ is accumulated by $C^2(b)$-segments stretching across Θ, for any $z \in \gamma$ it is possible to define its bound/free structure. Suppose that $x, y \in \gamma$, $n > 0$ are such that $f^n x$ is bound and $f^n y \in I(\delta)$. Then $f^n x$ is near Q, and thus $f^{n+1}\gamma$ intersects both R_0 and R_1. This yields a contradiction. Hence, it follows that if $x \in \gamma$, $n > 0$ and $f^n x$ is bound, then $f^n\gamma \cap I(\delta) = \emptyset$. Then it is possible to take an arbitrarily large integer n such that all points on $f^n\gamma$ are free, and thus $\text{length}(f^n\gamma) \geq 10^{-10}\lambda_1^{\frac{n}{3}} \cdot \text{length}(\gamma)$. This implies that some forward iterates of γ intersect both R_0 and R_1, a contradiction. $\square$

For $n \in \mathbb{Z}$, let $A_n(\omega) = \{x \in \pi(\omega) \colon f^n x \in \Theta\}$.

Lemma 3.6. *The following statements hold:*
(a) $A_n(\omega)$ consists of at most one point;
(b) For any m, n, either (i) $A_m(\omega) = A_n(\omega)$, or (ii) $A_m(\omega) = \emptyset$ or $A_n(\omega) = \emptyset$.

Proof. We have $f^n A_n(\omega) = \Theta \cap \pi(\sigma^n\omega)$. Hence Lemma 3.4 gives (a). To prove (b) we need

Sublemma 3.7. *If $x \in R \setminus \Theta$ and $y \in \Theta$, then $\pi^{-1}(x) \neq \pi^{-1}(y)$.*

If (i) (ii) do not hold, then there exists $x \in A_m(\omega)$ and $y \in A_n(\omega)$ such that $f^m(x) \in \Theta$ and $f^m(y) \notin \Theta$. Sublemma 3.7 gives $\pi^{-1}(f^m A_m(\omega)) \neq \pi^{-1}(f^m A_n(\omega))$. This yields a contradiction to the definition of $\pi(\omega)$.

It is left to prove the sublemma. For $x \in K$ and $n \in \mathbb{Z}$, define $\omega_n(x) \in \{0,1\}$ by $f^n x \in R_{\omega_n(x)}$. We claim that if $x \in R \setminus \Theta$ and $y \in \Theta$, then there exists $n \in \mathbb{N}$ such that $\omega_n(x) \neq \omega_n(y)$. To see this, define rectangles S_1, S_2, S_3, S_4 as follows: S_1 (resp. S_4) is the component of $R \setminus \mathrm{Int}\Theta$ at the left (resp. right) of ζ_0; $S_2 = R_0 \setminus \mathrm{int} S_1$ and $S_3 = R_1 \setminus \mathrm{int} S_4$ (See Figure 5). Observe that $f(S_1) \subseteq S_1 \cup S_2 \cup S_3$ and $f(S_2) \subseteq S_4$ and $f(S_3) \subseteq S_4$ and $f(S_4) \subseteq S_1 \cup S_2 \cup S_3$. Either: (i) $x \in S_1$, $y \in S_2$; (ii) $x \in S_1$, $y \in S_3$; (iii) $x \in S_4$, $y \in S_2$; (iv) $x \in S_4$, $y \in S_3$. In cases (ii) and (iii) we have $\omega_0(x) \neq \omega_0(y)$ so the sublemma holds for $n = 0$. In case (i), either $\omega_0(x)\omega_1(x) = 00$ while $\omega_0(y)\omega_1(y) = 01$ and the sublemma holds for $n = 2$, or else $fx \in S_3$ and $fy \in S_4$ as in case iv). In case (iv), either $\omega_0(x)\omega_1(x) = 10$ while $\omega_0(y)\omega_1(y) = 11$and the sublemma holds for $n = 2$, or else $fx \in S_3 \cap f(R)$ and $fy \in S_4 \cap f(R)$.

If $fx \in I(\delta) \cap S_3$ then $\omega_0(x)\omega_1(x)\omega_2(x)\omega_3(x) = 1110$ while $\omega_0(y)\omega_1(y)\omega_2(y)\omega_3(y) = 1100, 1101$ or 1111. In the first two cases the sublemma holds for $n = 3$. Let us now assume that $fx \in S_3 \setminus I(\delta)$. Given any b-horizontal curve γ_0 containing fx, let $\xi := \gamma_0 \cap \alpha_1^+$. Since $f(\alpha_1^+) \subseteq \alpha_1^+$, Lemma 2.3 implies that the horizontal distance between $f^i x$ and α_1^+ grows exponentially as long as the iterates of $f^i x \in R_1 \setminus I(\delta)$. Hence there exists some n for which $f^{n-1}x \in S_4$ and $f^n x \in R_0$. Since $f(S_3) \subseteq S_4$ and $f(S_4) \cap S_4 = \emptyset$ then either $\omega_j(y) = 0$ for some $j < n$ or $\omega_n(y) = 1$. This proves the sublemma. $\square$

Let $E = \{x \in R \cap K \colon f^n z \notin \Theta \ \forall n \in \mathbb{Z}\}$. We have

$$\pi(\omega) = \{x \in E \colon \pi^{-1}(x) = \omega\} \cup \bigcup_n A_n(\omega). \tag{20}$$

It is easy to see that E is contained in the stable sides of R. In addition, Lemma 2.2 implies that if $x, y \in K$ belong to the same stable side of R, then $\pi^{-1}(x) \neq \pi^{-1}(y)$. Hence the first set in (20) consists of at most one point. By Lemma 3.6, the second set in (20) consists of at most one point. One of these two sets are empty, for otherwise Sublemma 3.7 yields a contradiction. Consequently, $\pi(\omega)$ consists of one point for any $\omega \in \Sigma_2$. Since $R_0 \cap R_1 = \{\zeta_0\}$, π is 1-1 except on $\bigcup_{i=-\infty}^{\infty} f^i \zeta_0$ where it is 2-1. Observe that, since σ sends cylinder sets to cylinder sets, the continuity of π at a point ω implies the continuity of π at $\sigma^n \omega$, $n \in \mathbb{Z}$. The continuity of π on $\pi^{-1}\Theta$ follow from the proof of Lemma 3.4. By the above observation, π is continuous on $\Sigma_2 \setminus \pi^{-1}E$. The continuity on $\pi^{-1}E$ is obvious. Since $K \subset R_0 \cup R_1$, π is surjective. This completes the proof of Proposition 3.1. $\square$

4. Proofs of the theorems

In this section we complete the proofs of the theorems. In Sect.4.1 we study the regularity of the unstable direction E^u defined in (2). In Sect.4.2 we estimate Lyapunov exponents of limit points in the weak topology. Finally we prove the Theorem in Sect.4.3.

4.1. **Regularity of the unstable direction.** For two positive integers i, j, let $K_{i,j}$ denote the set of all $z \in K$ such that there exists $v \in T_z\mathbb{R}^2$ such that $\|Df^{-n}(z)v\| \leq ij^{-n}\|v\|$ holds for $n \geq 0$. Then $K_{i,j}$ is a closed set. Observe that $E^u(z)$ makes sense if and only if there exist i, j such that $z \in K_{i,j}$. Since E^u is continuous on $K_{i,j}$, it is Borel measurable on $\bigcup_{i,j} K_{i,j}$.

Due to the presence of the tangency, E^u is not continuous at Q. We show that E^u makes sense and is continuous on a large subset[1] of K. Let $\partial^s R$ denote the union of the stable sides

[1]We do not prove that E^u is well-defined or continuous on $\partial^s R \setminus \{Q\}$. This does not matter because any f-invariant measure gives zero weight to this set.

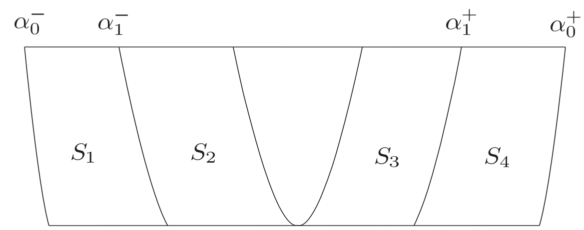

FIGURE 5.

of R and let $K' = (K \setminus \partial^s R) \cup \{Q\}$. Observe that any f-invariant Borel probability measure gives full weight to K'.

Proposition 4.1. *E^u is well-defined on K', and is continuous on $K' \setminus \{Q\}$.*

Proof. We first prove that E^u makes sense on $\mathcal{W}^u$, and is spanned by the tangent directions of the unstable leaves in Γ^u. Since any unstable leaf is a C^1 limit of a sequence of curves in $\tilde\Gamma^u$, these statements follow from the next uniform backward contraction on curves in $\tilde\Gamma^u$.

Lemma 4.2. *For any $\gamma \in \tilde\Gamma^u$, $z \in \gamma$ and $n > 0$, $\|Df^n|E^u(f^{-n}z)\| \geq \frac{1}{10}\lambda_1^{\frac{n}{2}}$.*

Proof. Take a large integer $M \geq n$ so that $f^{-M}z$ is contained in the local unstable manifold of the saddle. We introduce a bound/free structure for the forward orbit of $f^{-M}z$. Observe that $z \in \Theta$ must be free, as the forward orbit of a critical point never returns close to Θ.

We first consider the case where $f^{-n}z$ is free. Splitting the orbit $f^{-n}z, f^{-n+1}z \ldots, z$ into bound and free segments, and then applying the derivative estimates in (19) we get the desired inequality.

We now consider the case where $f^{-n}z$ is bound. By definition, there exist $i > n$ and a critical point ζ relative to which $f^{-i}z$ is in tangential position. Let p, q denote the corresponding bound and fold periods. We have $-n < -i + p$. There are two cases, $-n$ being either inside or outside of the fold period. If $-n < -i + q$, then

$$\|Df^n|E^u(f^{-n}z)\| = \frac{\|Df^i|E^u(f^{-i}z)\|}{\|Df^{i-n}|E^u(f^{-i}z)\|} \geq \|Df^i|E^u(f^{-i}z)\| \geq \lambda_1^{\frac{i}{2}} > \lambda_1^{\frac{n}{2}}.$$

For the first inequality we have used (d) Proposition 2.6. If $-n \geq -i + q$, then (c) Proposition 2.6 and equation (4) give

$$\frac{\|Df^p|E^u(f^{-i}z)\|}{\|Df^{i-n}|E^u(f^{-i}z)\|} \geq \frac{1}{10}\frac{\|w_p(\zeta)\|}{\|w_{i-n}(\zeta)\|} \geq \frac{1}{10}\lambda_2^{p-i+n}.$$

Since both $f^{p-i}z$ and z are free, Proposition 2.6 and Lemma 2.3 yield

$$\|Df^i|E^u(f^{-i}z)\| \geq \lambda_1^{\frac{i-p}{2}}\|Df^p|E^u(f^{-i}z)\|.$$

Multiplying these two inequalities and then using $n > i - p$ we obtain

$$\|Df^n|E^u(f^{-n}z)\| \geq \frac{1}{10}\lambda_2^{p-i+n}\lambda_1^{\frac{i-p}{2}} \geq \frac{1}{10}\lambda_2^{\frac{1}{2}(p-i)}\lambda_1^{n} \geq \frac{1}{10}\lambda_1^{\frac{n}{2}}.$$ □

We have proved that E^u is defined on $\mathcal{W}^u$. By Lemma 2.9, E^u is defined everywhere on $\Theta \cap K$. Due to the Df-invariance, E^u is defined everywhere on $\bigcup_{n=-\infty}^{0} f^n(\Theta \cap K)$. Obviously we have $K \setminus \bigcup_{n=-\infty}^{0} f^n(\Theta \cap K) \subset \partial^s R$, and thus E^u is well-defined on K'.

Lemma 2.2 implies that E^u is uniformly continuous on $\Theta \cap W^u$, and thus it is continuous on $\mathcal{W}^u$. Let $z \in K' \setminus \{Q\}$. Then there exists $n \geq 0$ such that $f^n z \in \Theta$. We first consider the case where $f^n z$ is not in the stable sides of Θ. Then E^u is continuous at $f^n z$, and so the Df-invariance of E^u and the continuity of Df together imply that E^u is continuous at z as well.

In the case where $f^n z$ is in the stable sides of Θ, the above argument is slightly incomplete, because the continuity of E^u in a neighborhood of $f^n z$ is not proved yet. However, we can prove this by slightly extending the region Θ and repeating the same arguments. □

4.2. **Unstable Lyapunov exponents of limit points.** A main result in this subsection is as follows. Let $\mathcal{M}^e(f)$ denote the set of all ergodic f-invariant probability measures and let δ_Q denote the Dirac measure at Q.

Proposition 4.3. *If* $(\mu_n) \subset \mathcal{M}^e(f) \setminus \{\delta_Q\}$, $\mu_n \to \mu$, $\mu = u\delta_Q + (1-u)\nu$, $0 \leq u \leq 1$, $\nu \in \mathcal{M}(f)$ *and* $\nu(\{Q\}) = 0$, *then:*

$$u\frac{\log \lambda_2}{3} + (1-u)\lambda^u(\nu) \leq \varliminf_{n\to\infty} \lambda^u(\mu_n);$$

$$\varlimsup_{n\to\infty} \lambda^u(\mu_n) \leq u\lambda^u(\delta_Q) + (1-u)\lambda^u(\nu).$$

Proof. For each $k \geq 1$, let $\tilde{V}_k$ denote the rectangle which is bordered by $\tilde{\alpha}_k$ and the boundaries of R and which does not contain ζ_0 (see Sect.2.1 for the definitions of $\tilde{\alpha}_k$). Set

$$V_k := \bigcup_{i=0}^{100k} f^i \tilde{V}_{101k}.$$

Observe that $\{V_k\}$ is a nested sequence, and $\bigcap_{k=1}^{\infty} V_k = \alpha_0^-$.

Fix a partition of unity $\{\rho_{0,k}, \rho_{1,k}\}$ on R such that

$$\mathrm{supp}(\rho_{0,k}) = \overline{\{x \in R\colon \rho_{0,k}(x) \neq 0\}} \subset V_k \quad \text{and} \quad \mathrm{supp}(\rho_{1,k}) \subset R \setminus \overline{V_{2k}}.$$

We argue with subdivision into two cases.

Case I: $u = 0$*.* The desired inequalities are direct consequences of the next

Lemma 4.4. *If* $\{\mu_n\}_n \subset \mathcal{M}(f)$, $\mu_n \to \mu$ *and* $\mu(\{Q\}) = 0$, *then* $\lambda^u(\mu_n) \to \lambda^u(\mu)$.

Proof. Set $\overline{L} = \varlimsup_{n\to\infty} \lambda^u(\mu_n)$ and $\underline{L} = \varliminf_{n\to\infty} \lambda^u(\mu_n)$. Taking subsequences if necessary we may assume $\overline{L} = \lim_{n\to\infty} \lambda^u(\mu_n)$. For all k,

$$\overline{L} = \lim_{n\to\infty} \mu_n(\rho_{0,k} \log J^u) + \lim_{n\to\infty} \mu_n(\rho_{1,k} \log J^u). \tag{21}$$

Let $\varepsilon > 0$. For sufficiently large k we have $\mu(V_k) \leq \varepsilon$. Since $\mu(\{Q\}) = 0$ and $\mu \in \mathcal{M}(f)$ we have $\mu(\partial V_k) = 0$, and thus $\mu_n(V_k) \to \mu(V_k)$ as $n \to \infty$. In particular, for sufficiently large n we have $\mu_n(V_k) \leq 2\varepsilon$, and therefore

$$\lim_{n\to\infty} \mu_n(\rho_{0,k} \log J^u) \leq \log 5 \cdot \varlimsup_{n\to\infty} \mu_n(V_k) \leq 2\varepsilon \log 5.$$

Since $\varepsilon > 0$ is arbitrary, it follows that the first term of the right-hand-side of (21) goes to 0 as $k \to \infty$.

By Proposition 4.1, $\rho_{1,k}\log J^u$ is continuous, and the weak convergence gives

$$\lim_{n\to\infty}\mu_n(\rho_{1,k}\log J^u)=\mu(\rho_{1,k}\log J^u).$$

From the Dominated Convergence Theorem, the second term of the right-hand-side of (21) goes to $\lambda^u(\mu)$ as $k\to\infty$. Hence we obtain $\overline{L}=\lambda^u(\mu)$. The same reasoning gives $\underline{L}=\lambda^u(\mu)$. $\square$

Case II: $u>0$.

Lemma 4.5. *There exists a large integer $k_0=k_0(b)$ such that the following holds for every $k\geq k_0$: if $z\in K$, $m>0$ are such that $f^{-1}z\notin V_k$, $z,fz,\ldots,f^{m-1}z\in V_k$, $f^mz\notin V_k$, then*

$$\frac{1}{3}\log\lambda_2\leq\frac{1}{m}\sum_{i=0}^{m-1}\log J^u(f^iz)\leq\lambda^u(\delta_Q).$$

Proof. Let $z\in K$, $m>0$ be as in the statement. If $z\in f^i\tilde{V}_{101k}$ holds for some $0<i\leq 100k$, then $f^{-1}z\in V_k$. Hence $z\notin f^i\tilde{V}_{101k}$ holds for every $0<i\leq 100k$. Since $z\in V_k$, we have $z\in\tilde{V}_{101k}$. Hence

$$m-1\geq 100k\ \text{ and }\ f^{-2}z\in I(\delta). \tag{22}$$

Let $y=f^{-2}z$. Let ζ denote the binding point for y and p the bound period. Let $v(y)$ denote any vector which spans $E^u(y)$. Split

$$Dfv(y)=A\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+B\cdot e^s(fy).$$

Observe that

$$\frac{1}{m}\sum_{i=0}^{m-1}\log J^u(f^iz)=\frac{1}{m}\log\frac{\|Df^{m+2}v(y)\|}{\|Df^2v(y)\|}.$$

In the sequel we argue as in the proof of Proposition 2.6. Choose a $C^2(b)$-curve γ which connects fy and $\mathcal{F}^s(f\zeta)$. We have length$(\gamma)\approx|\zeta-y|^2$. Since $f^i\gamma$ $(i=0,1,\ldots,m+2)$ are $C^2(b)$-curves located outside of Θ, and $f^{m+1}y\in V_k$, $f^{m+2}y\notin V_k$, for some $C\geq 1$ we have

$$C^{-1}\cdot{\lambda_3}^{-k}\leq\text{length}(f^{m+2}\gamma)\leq C\cdot{\lambda_2}^{-k+1}. \tag{23}$$

The bounded distortion gives

$$|\zeta-y|^2\cdot\|w_{m+2}(\zeta)\|\approx\text{length}(f^{m+2}\gamma). \tag{24}$$

From the proof of Proposition 2.6, there exists $C\geq 1$ such that

$$C^{-1}(\lambda_3+\varepsilon)^{-\frac{p}{2}}\leq|\zeta-y|\leq C\lambda_2^{-\frac{p}{2}}. \tag{25}$$

Using (23) (24) (25), for some $C>0$ we have

$$|A|\cdot\|Df^{m+1}(fy)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|\approx|\zeta-y|\cdot\|w_{m+2}(\zeta)\|\geq C\cdot{\lambda_3}^{-k}|\zeta-y|^{-1}\geq C\cdot{\lambda_3}^{-k}\lambda_2^{\frac{p}{2}}.$$

For the other component in the splitting we have

$$|B|\cdot\|Df^{m+1}e^s(fy)\|\leq(Cb)^{m+1}\leq(Cb)^m.$$

Since $\|Df^2v(y)\|\leq 1$, if $p\geq m+2$ then using (22) we have

$$\frac{\|Df^{m+2}v(y)\|}{\|Df^2v(y)\|}\geq C\cdot{\lambda_3}^{-k}\lambda_2^{\frac{p}{2}}-(Cb)^m\geq\lambda_2^{\frac{m}{3}}.$$

In the case $p < m+2$ the same inequality follows from Proposition 2.6(e). This yields the first inequality of the lemma. The second one is obvious. $\square$

Returning to the proof of Proposition 4.3 in the case $u > 0$, take a generic point ξ_n with respect to μ_n. For any $\eta > 0$, there exists n_0 such that for all $n \geq n_0$, $\mu_n(V_k) \geq u - \eta > 0$. Approximating the indicator function of V_k by a continuous bump function we have $\lim_{m\to\infty} \frac{1}{m}\#\{0 \leq i < m \colon f^i\xi_n \in V_k\} \geq u - \eta$. Since $\mu_n \neq \delta_Q$, the positive orbit of ξ_n is a concatenation of segments in V_k and those out of V_k. Then Lemma 4.5 gives

$$\mu_n(\rho_{0,k}\log J^u) = \lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}(\rho_{0,k}\log J^u(f^i(\xi_n)) \geq (u-\eta)\frac{\log\lambda_2}{3}.$$

If $u < 1$, then the weak convergence for the sequence $(\frac{\mu_n - u\delta_Q}{1-u})_n \subset \mathcal{M}(f)$ implies

$$\lim_{n\to\infty}\mu_n(\rho_{1,k}\log J^u) = (1-u)\nu(\rho_{1,k}\log J^u).$$

The same inequality remains true in the case $u = 1$. As $\eta > 0$ was chosen arbitrary we have

$$\varliminf_{n\to\infty}\lambda^u(\mu_n) \geq \varliminf_{n\to\infty}\mu_n(\rho_{0,k}\log J^u) + \lim_{n\to\infty}\mu_n(\rho_{1,k}\log J^u) \geq u\frac{\log\lambda_2}{3} + (1-u)\nu(\rho_{1,k}\log J^u).$$

Since $\nu(\{Q\}) = 0$, $\rho_{1,k}\log J^u \to \log J^u$ ν-a.e. as $k\to\infty$. Letting $k\to\infty$ and then using the Dominated Convergence Theorem gives the first estimate in the proposition. A proof of the second one is completely analogous. $\square$

4.3. **Existence of t-conformal measures.** We now complete the proof of the theorem.

Proof of the theorem. By the ergodic decomposition theorem [14], the unstable Lyapunov exponent of μ is written as a linear combination of the unstable Lyapunov exponents of its ergodic components. Since the same property holds for entropies, there exists an ergodic component μ' of μ such that $F_{\varphi_t}(\mu') = F_{\varphi_t}(\mu)$. Hence $P(t) = \sup\{F_{\varphi_t}(\mu) \colon \mu \in \mathcal{M}^e(f)\}$. Choose a convergent sequence $(\mu_n) \subset \mathcal{M}^e(f)\setminus\{\delta_Q\}$ such that $F_{\varphi_t}(\mu_n) > P(t) - \frac{1}{n}$. Let $\mu \in \mathcal{M}(f)$ denote the limit point. In the case $t \leq 0$, the upper semi-continuity of entropy and Proposition 4.3 yield $P(t) = \lim_{n\to\infty} F_{\varphi_t}(\mu_n) \leq F_{\varphi_t}(\mu) \leq P(t))$, namely μ is a t-conformal measure.

We now consider the case $t > 0$. Write $\mu = u\delta_Q + (1-u)\nu$ where $0 \leq u \leq 1$, $\nu \in \mathcal{M}(f)$ and $\nu(\{Q\}) = 0$. The upper semi-continuity of entropy gives

$$P(t) = \lim_{n\to\infty} F_{\varphi_t}(\mu_n) \leq h(\mu) - t\varliminf_{n\to\infty}\lambda^u(\mu_n).$$

If $u = 1$ then $\mu = \delta_Q$ and thus $h(\mu) = 0$. Proposition 4.3 gives $P(t) \leq -t\frac{\log\lambda_2}{3}$ and a contradiction arises because t is such that $P(t) > -t\frac{\log\lambda_2}{3}$. Hence $u < 1$ holds. If $u > 0$, then using Proposition 4.3 and $h(\mu) = (1-u)h(\nu)$ we have

$$\begin{aligned} P(t) &\leq h(\mu) - t\left(u\frac{\log\lambda_2}{3} + (1-u)\lambda^u(\nu)\right) \\ &= (1-u)F_{\varphi_t}(\nu) - tu\frac{\log\lambda_2}{3} < (1-u)F_{\varphi_t}(\nu) + uP(t). \end{aligned}$$

Rearranging this gives $(1-u)P(t) < (1-u)F_{\varphi_t}(\nu)$, and thus $P(t) < F_{\varphi_t}(\nu)$, a contradiction. Hence $u = 0$ holds, and $P(t) \leq F_{\varphi_t}(\nu) = F_{\varphi_t}(\mu)$, namely ν is a t-conformal measure. $\square$

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